

# Analytical Pricing of American Bond Options in the Heath-Jarrow-Morton Model\*

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## Abstract

In this paper we study the optimal stopping problem of pricing an American Put option on a Zero Coupon Bond (ZCB) in the Heath-Jarrow-Morton (HJM) framework for the forward interest rate. In particular we consider its Musiela's parametrization to guarantee a Markovian setting. Hence we are in an infinite dimensional setting, in which the forward rate curve is described by a SDE in a suitable Hilbert space.

In order to find an infinite dimensional variational formulation of the pricing problem, we extend some results on infinite dimensional optimal stopping and variational inequalities recently obtained in [8]. The proof goes through three main steps. First we regularize the American bond option's payoff by adopting usual smoothing arguments. Next we approximate the infinite dimensional dynamics by finite dimensional ones to which we associate suitable optimal stopping problems in  $\mathbb{R}^n$ . Then, by taking the limit as  $n \rightarrow \infty$  and by removing the smoothing on the payoff, we obtain an infinite dimensional variational inequality for the price of the American bond option. Moreover, the first time at which the price of the American bond option equals the payoff turns out to be an optimal exercise time.

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**Key words:** pricing American bond options, HJM model, forward interest rates, Musiela's parametrization, optimal stopping, infinite-dimensional stochastic analysis.

## 1 Introduction

A major challenge in mathematical finance is pricing derivatives with an increasing degree of complexity. A huge theoretical effort has been made in the last forty years to provide suitable tools for this purpose. The volume of traded options and the wide variety of their structures require a deep analysis of both theoretical and numerical methods.

An important class of traded options is that of *American options*. The mathematical formulation of this problem was given in the eighties by A. Bensoussan [2] and I. Karatzas [26], among others. In mathematical terms pricing an American option corresponds to solving an *optimal stopping* problem (for a good survey cf. [34]) in which the state dynamics is that of the security underlying the contract, usually a diffusion process (cf. Jacka [24] for a geometric Brownian motion and [25] for more general diffusions). In such case one may find a variational

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formulation of the optimal stopping problem; that is, a free-boundary problem in the language of PDE (cf. for instance [3], [19] and [27] for a survey).

Here we aim to study the problem of pricing an American Put option on a Zero Coupon Bond (American bond option) with the forward interest rate process as underlying. This option gives the holder the right to sell the ZCB for a fixed price  $K$  at any time prior to the maturity  $T$ . The forward rate is the instantaneous interest rate agreed at time  $t$  for a loan which will take place at a future time  $T \geq t$ . It is often denoted by  $f(t, T)$  and taking  $T = t$  one recovers the “so called” spot rate  $R(t) = f(t, t)$ . The price of the Bond,  $B(t, T)$ , is linked to the forward rate by the ordinary differential equation

$$f(t, T) = -\frac{\partial}{\partial T} \ln(B(t, T)). \quad (1.1)$$

The option payoff at time  $t$  is given by  $[K - B(t, \hat{T})]^+$ , where  $[\cdot]^+$  denotes the positive part and  $\hat{T} \in [T, T_{\max}]$  is the bond maturity. The arbitrage free price of the American bond option is defined as

$$V(t, f(t, \cdot)) = \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ e^{-\int_t^\tau R(s) ds} [K - B(\tau, \hat{T})]^+ \right\}. \quad (1.2)$$

Notice that  $V$  depends on the entire forward curve as it is typical of infinite dimensional optimization problems; hence one expects that it should solve an infinite dimensional variational inequality. However, for American options with an infinite dimensional underlying process it is not straightforward to establish a connection with PDE's in Hilbert spaces (cf. [10], [13]). Such connection is instead known for *European options* under forward rates; in fact their prices may be uniquely characterized through specific Kolmogorov equations (cf. [21]). In some sense, that is a natural generalization of the Black and Scholes pricing formula to the infinite dimensional setting. Infinite dimensional variational inequalities have not received as much attention as their finite dimensional counterparts. A good survey may be found in [1], [8], [9], [20], [35], [36] and the references therein.

There exists a large literature on interest rate models concerning both theoretical and numerical aspects (for good surveys cf. [4], [6], [32] for instance). In this paper, for the forward interest rates we choose the framework of the famous HJM model, one of the most reliable ones, which was introduced by D. Heath, R. Jarrow and A. Morton [23] in 1992. The peculiarity of the stochastic process representing the forward interest rate is its infinite dimensional character. In essence, at each time  $t$ , the HJM model describes the family of rates  $f(t, T)$ , with  $T \in [t, T_{\max}]$ , that is the whole term structure of forward rates. A suitable parametrization of  $f(t, T)$ , modeled by an infinite dimensional stochastic differential equation, was obtained by M. Musiela [31] in 1993. An exhaustive description of the HJM model and its offspring may be found in [17] and [18].

Our financial problem has been studied in [20] by means of viscosity theory, although in a different framework; that is, under the Goldys-Musiela-Sondermann parametrization ([22]) of the HJM model. That completely determines the volatility structure of the dynamics and simplifies the underlying infinite dimensional stochastic differential equation by removing an unbounded term in the drift. A possible drawback of the model in [20] is the lack of consistency with the market's observations. This fact has been extensively discussed by D. Filipovic in [17].

It is worth mentioning that an attempt to provide some numerical results for the price function  $V$  was recently made in [29]. However in that paper arguments are mostly heuristic, proofs are only sketched and some of them seem to be incorrect.

We provide a variational formulation of the pricing problem (1.2) which is the infinite dimensional extension of that in [25]. Ours is (in some sense) a generalization of [3]. We also find an optimal exercise time for the American Bond option. Our approach is based on [8].

The paper is organized as follows. In Section 2 we introduce the financial model of the forward interest rate dynamics. In Section 3 we give the mathematical formulation of the corresponding optimal stopping problem and we obtain some regularity properties of the American bond option's price  $V$ . Section 4 is devoted to a regularization of the Put payoff  $\Psi$ . We associate an optimal stopping problem with value function  $V_k$  to each smooth approximation  $\Psi_k$  of the original payoff  $\Psi$ . Then we show that  $V_k \rightarrow V$  as  $k \rightarrow \infty$ . In Section 5 we approximate the infinite dimensional optimal stopping problem  $V_k$  by a sequence of finite dimensional ones. By using arguments as in [8] we prove that  $V_k$  is a suitable solution of an infinite dimensional variational inequality. Section 6 contains the main result of this paper. There we obtain an infinite dimensional variational inequality for the price  $V$  of the original American bond option. Also, we show that the first time at which  $V$  equals the payoff  $\Psi$  is an optimal exercise time for the option's holder. A technical appendix completes the paper.

## 2 The interest rate model

The forward rate at time  $t$  for a loan taking place at a future time  $T \geq t$  and returned at  $T + dT$  is commonly denoted by  $f(t, T)$ . The instantaneous spot rate is obtained by setting  $T = t$  and it is denoted by  $R(t) := f(t, t)$ . For every fixed maturity  $T$  the time evolution of the forward rate is described by the map  $t \mapsto f(t, T)$  with  $t \leq T$ .

Consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and denote by  $(\mathcal{F}_t)_{0 \leq t \leq T}$  the filtration generated by a  $d$ -dimensional Brownian motion  $B$ , completed with the null sets. For simplicity but with no loss of generality we take  $d = 1$ . Let  $C_b^{0,1}(\mathbb{R})$  denote the set of bounded, Lipschitz-continuous real functions. Take  $\tilde{\sigma} \in C_b^{0,1}(\mathbb{R})$ ,  $\tilde{\sigma}$  non-negative. According to the Heath-Jarrow-Morton model (HJM) (cf. [23]),  $\mathbb{P}$  may be assumed to be the risk-neutral probability measure on  $\Omega$  and the forward rate with maturity  $T$  may be described by the SDE

$$f(t, T) = f(0, T) + \int_0^t \tilde{\sigma}(f(u, T)) \int_u^T \tilde{\sigma}(f(u, s)) ds du + \int_0^t \tilde{\sigma}(f(u, T)) dB_u, \quad t \in [0, T], \quad (2.1)$$

where  $f(0, T)$  is deterministic and denotes the initial data at time zero.

Notice that for the purposes of this work there is no substantial loss of generality if we assume time-homogeneity for the coefficient  $\tilde{\sigma}$  in (2.1). Under such assumption there exists a unique strong solution  $f(\cdot, \cdot)$  continuous in both variables (cf. [30]). Unfortunately the process  $\{f(t, T)\}_{0 \leq t \leq T}$  is not Markovian since the drift in (2.1) depends on the evolution of the whole forward curve. On the other hand, the Markov property holds for the infinite dimensional process  $t \mapsto \{f(t, v), t \leq v \leq T\}$ ; therefore, pricing derivatives often requires to set dynamics in the infinite dimensional SDE's framework (cf. [12]). This is accomplished by means of the so-called Musiela's parametrization (cf. [31]) that describe the forward rate curve  $f(t, T)$  in terms of the time to maturity  $x := T - t$  rather than the maturity time  $T$ ; hence  $f(t, T) = f(t, t + x)$ . Then, in terms of the original forward curve we define the map  $(t, x) \mapsto r_t(x)$  by setting  $r_t(x) := f(t, t + x)$ ; that is, at any given time  $t$  the model's input is the forward rate curve  $x \mapsto r_t(x)$ . The short rate is obtained by taking  $x = 0$  and it is denoted by  $r_t(0)$ .

The process  $t \mapsto r_t(\cdot)$  may be interpreted as an infinite-dimensional process taking values in a suitable Hilbert space  $\mathcal{H}$ . On such space the unbounded linear operator  $A := \frac{\partial}{\partial x}$  generates a  $C_0$ -semigroup of bounded linear operators  $\{S(t) \mid t \in \mathbb{R}_+\}$ . In particular,  $S(\cdot)$  is the semigroup of right shifts defined by  $S(t)h(x) = h(t + x)$  for any function  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  (for further details on semigroup theory the reader may refer to [33]). Define  $\sigma(r_t)(x) := \tilde{\sigma}(f(t, T))$  and set

$$F_\sigma(r_t)(x) := \sigma(r_t)(x) \int_0^x \sigma(r_t)(y) dy, \quad x \in \mathbb{R}_+. \quad (2.2)$$

Long but straightforward calculations allow to write (2.1) as

$$r_t(x) = S(t)r_0(x) + \int_0^t S(t-u)F_\sigma(r_u)(x)du + \int_0^t S(t-u)\sigma(r_u)(x)dB_u. \quad (2.3)$$

The link to the theory of infinite dimensional SDE's is now rather natural; in fact, under appropriate conditions on  $\sigma$  and  $\mathcal{H}$ , (2.3) turns out to be the unique *mild* solution of

$$\begin{cases} dr_t = [Ar_t + F_\sigma(r_t)]dt + \sigma(r_t)dB_t, & t \in [0, T], \\ r_0 = r \in \mathcal{H}, \end{cases} \quad (2.4)$$

where  $0 < T < \infty$  (cf. [12], Chapter 7).

In the present work  $\mathcal{H}$  is chosen according to [17] and the notation  $\mathcal{H} = \mathcal{H}_w$  is adopted (other possible models are available in [21] and [22], among others). Some fundamental facts are recalled in what follows.

**Definition 2.1.** Let  $w : \mathbb{R}_+ \rightarrow [1, +\infty)$  be a non decreasing  $C^1$ -function such that

$$w^{-\frac{1}{3}} \in L^1(\mathbb{R}_+). \quad (2.5)$$

Define

$$\mathcal{H}_w := \{h \in L^1_{loc}(\mathbb{R}_+) \mid \exists h' \in L^1_{loc}(\mathbb{R}_+) \text{ and } \|h\|_w < \infty\}, \quad (2.6)$$

where

$$\|h\|_w^2 := |h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 w(x) dx. \quad (2.7)$$

The derivatives in Definition 2.1 are weak derivatives and the space  $(\mathcal{H}_w, \|\cdot\|_w)$  is a Hilbert space (cf. [17], Theorem 5.1.1). An important consequence of (2.5) and (2.7) is the continuous injection  $\mathcal{H}_w \hookrightarrow L^\infty(\mathbb{R}_+)$  (cf. [17], Chapter 5, Eq. 5.4), i.e. there exists  $C > 0$  such that

$$\sup_{x \in \mathbb{R}_+} |h(x)| \leq C \|h\|_w, \quad h \in \mathcal{H}_w. \quad (2.8)$$

Also, we point out that if  $h \in \mathcal{H}_w$  then  $\lim_{x \rightarrow \infty} h(x)$  exists and is finite (cf. [17], p. 77).

Recall that  $\tilde{\sigma} \in C_b^{0,1}(\mathbb{R})$  and hence  $\sigma$  inherits the same regularity as a function  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ . However in (2.4) one must think of  $\sigma$  and  $F_\sigma$  as functions:  $\mathcal{H}_w \rightarrow \mathcal{H}_w$ . Define the set

$$\mathcal{H}_w^0 := \{h \in \mathcal{H}_w \mid h(\infty) = 0\}, \quad (2.9)$$

then the following proposition holds (cf. [17], Chapter 5, Eq. (5.13)).

**Proposition 2.2.**  $\mathcal{H}_w^0$  is a closed subspace of  $\mathcal{H}_w$ . Moreover,  $F_\sigma$  takes values in  $\mathcal{H}_w$  if and only if  $\sigma$  takes values in  $\mathcal{H}_w^0$ .

As a consequence of Proposition 2.2 the only volatility structures allowed in (2.4) are those such that  $\sigma(h)(x) \rightarrow 0$  when  $x \rightarrow \infty$  for any  $h \in \mathcal{H}_w$ . From now on we will make the following Assumption.

**Assumption 2.3.** The volatility  $\sigma : \mathcal{H}_w \rightarrow \mathcal{H}_w^0$  is bounded and uniformly Lipschitz; i.e

$$\|\sigma(h)\|_w < C_\sigma \quad \text{and} \quad \|\sigma(f) - \sigma(h)\|_w \leq L_\sigma \|f - h\|_w \quad \text{for all } f, h \in \mathcal{H}_w \quad (2.10)$$

and for some positive constants  $C_\sigma$  and  $L_\sigma$ .

A simple extension of [17], Corollary 5.1.2, gives the following

**Proposition 2.4.** *Under Assumption 2.3 there exists  $L_F > 0$  depending on  $C_\sigma$ ,  $L_\sigma$  and such that*

$$\|F_\sigma(f) - F_\sigma(h)\|_w \leq L_F \|f - h\|_w, \quad \text{for } f, h \in \mathcal{H}_w. \quad (2.11)$$

Now the main results of [17], Chapter 5, may be summarized in the following

**Theorem 2.5.** *Let  $\mathcal{H}_w$  be as in (2.6),*

- (i) *each function  $h \in \mathcal{H}_w$  has a continuous representative and the pointwise evaluation  $\mathcal{J}_x(h) := h(x)$  is a continuous linear functional on  $\mathcal{H}_w$ , for all  $x \in \mathbb{R}_+$ ;*
- (ii) *the semigroup  $\{S(t) | t \in \mathbb{R}_+\}$  is strongly continuous in  $\mathcal{H}_w$  with infinitesimal generator denoted by  $A$ , where  $D(A) = \{h \in \mathcal{H}_w | h' \in \mathcal{H}_w\}$  and  $Ah = h'$ .*

Moreover, under Assumption 2.3 there exists a constant  $K > 0$  such that

$$\|F_\sigma(h)\|_w \leq K C_\sigma^2, \quad \text{for all } h \in \mathcal{H}_w. \quad (2.12)$$

The existence and uniqueness of the solution of (2.4) now follow.

**Theorem 2.6.** *Under Assumption 2.3 there exists a unique mild solution of (2.4).*

*Proof.* The proof follows by standard arguments (cf. [12], Theorem 7.4 and Theorem 6.5 and [17], Theorem 2.4.1) since  $F_\sigma$  is bounded and uniformly Lipschitz by (2.12) and Proposition 2.4.  $\square$

The next Lemma provides standard estimates for the solution.

**Lemma 2.7.** *Let  $r^h$  and  $r^g$  be the mild solutions of (2.4) starting at  $h$  and  $g$ , respectively. Then*

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq T} \|r_t^h\|_w^p \right\} \leq C_{p,T} (1 + \|h\|_w^p), \quad 1 \leq p < \infty, \quad (2.13)$$

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq T} \|r_t^h - r_t^g\|_w^p \right\} \leq C_{p,T} \|h - g\|_w^p, \quad 1 \leq p < \infty, \quad (2.14)$$

where the positive constant  $C_{p,T}$  depends only on  $p$  and  $T$ .

*Proof.* The proof of (2.13) follows from [12], Theorem 7.4, whereas the proof of (2.14) is a consequence of [12], Theorem 9.1 and a simple application of Jensen's inequality.  $\square$

We now add some supplementary assumptions needed in the rest of the paper. We start by defining a trace class operator which will play a crucial role.

**Definition 2.8.** *Let  $Q : \mathcal{H}_w \rightarrow \mathcal{H}_w$  be the positive, linear operator defined by*

$$Q\varphi_i = \lambda_i \varphi_i, \quad \lambda_i > 0, \quad i = 1, 2, \dots,$$

and such that  $\sum_{i=1}^\infty \lambda_i < \infty$  (i.e. it is of trace class).

Let  $C_b^k(\mathcal{H}_w; \mathcal{H}_w^0)$  be the set of continuous, bounded functions  $f : \mathcal{H}_w \rightarrow \mathcal{H}_w^0$  with bounded and continuous Frechét derivatives up to order  $k$ . With no loss of generality we make the following Assumptions.

**Assumption 2.9.** *The operator  $Q$  of Definition 2.8 is such that*

$$\sum_{j=1}^\infty \|A\varphi_j\|_w \sqrt{\lambda_j} < \infty. \quad (2.15)$$

**Assumption 2.10.** *The map  $\sigma : \mathcal{H}_w \rightarrow \mathcal{H}_w^0$  is continuous and*

- (1)  $\sigma(h) \in Q(\mathcal{H}_w)$ ,  $\forall h \in \mathcal{H}_w$ ; i.e., there exists  $\gamma : \mathcal{H}_w \rightarrow \mathcal{H}_w^0$  such that  $\sigma(h) = Q\gamma(h)$ .
- (2)  $\gamma \in C_b^2(\mathcal{H}_w; \mathcal{H}_w^0)$ .

Notice that Assumption 2.9 implies that  $Q^{\frac{1}{2}}(\mathcal{H}_w) \subset D(A)$ , which together with Assumption 2.10 implies that  $\sigma$  maps  $\mathcal{H}_w$  into a subspace of  $D(A^2)$ . Also, whenever  $\lim_{x \rightarrow \infty} Q\gamma(h)(x) = \ell \neq 0$  (the limit always exists), one finds a volatility  $\sigma$  satisfying Assumption 2.10 with  $\tilde{\gamma}(h) = \gamma(h) - Q^{-1}\ell$ .

### 3 The pricing problem and some estimates

In terms of the unique solution of (2.4), the price at time  $t$  of a Zero Coupon Bond (ZCB) with maturity  $T \geq t$  may be expressed by

$$B(t, T; r_t(\cdot)) := \exp \left( - \int_0^{T-t} r_t(x) dx \right). \quad (3.1)$$

Recall that  $r.(0)$  is the spot rate, then the stochastic discount factor  $\beta$  at time  $t$  is

$$\beta(t; r.(0)) := \exp \left( - \int_0^t r_s(0) ds \right). \quad (3.2)$$

For  $t \geq s$  also define

$$D(s, t; r.(0)) := \frac{\beta(t; r.(0))}{\beta(s; r.(0))} = \exp \left( - \int_s^t r_u(0) du \right). \quad (3.3)$$

The HJM model allows non-negative forward rate curves for any finite maturity  $T > 0$  (cf. [23], Proposition 5), however in order to guarantee positive rates one has to further restrict the class of allowed volatilities<sup>1</sup>. Here we consider a modified discount factor that prevents the spot rate to go negative at any time, i.e.

$$D^+(s, t; r.(0)) = \exp \left( - \int_0^t [r_s(0)]^+ ds \right), \quad (3.4)$$

with  $[x]^+ = \max\{x, 0\}$ . As for the pricing problem, this choice does not spoil the results (cf. for instance [36], Section 4.2).

Assume that the forward rate curve at time  $t \in [0, T]$  is described by a function  $h \in \mathcal{H}_w$ , then the gain function at time  $t$  of the American Put option with strike price  $K < 1$  is

$$\Psi(t, h(\cdot)) := [K - B(t, T; h)]^+ = \left[ K - e^{-\int_0^{T-t} h(x) dx} \right]^+. \quad (3.5)$$

Let  $r_s^{t,r}$ ,  $s \geq t$  denote the value at time  $s$  of the solution of (2.4) with starting time  $t$  and initial data  $r$ . The value function  $V$  of the option evaluated at time  $t \leq T$  may be written under the risk-neutral probability measure  $\mathbb{P}$  as

$$V(t, r) := \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ e^{-\int_t^\tau [r_u^{t,r}(0)]^+ du} \left[ K - e^{-\int_0^{T-\tau} r_\tau^{t,r}(x) dx} \right]^+ \right\}. \quad (3.6)$$

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<sup>1</sup>The issue of forward rate's sign is hard to pose due to the infinite-dimensional character of the diffusion. It remains unsolved in most models relying on Musiela's parametrization of the dynamics.

In what follows it will be convenient to write (3.6) in terms of (3.1) and (3.4) as

$$V(t, r) = \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ D^+(t, \tau; r^{t,r}(0)) \Psi(\tau, r_\tau^{t,r}(\cdot)) \right\}. \quad (3.7)$$

Observe that the option pricing problem is meaningful only when the maturity of the option is lesser or equal than the maturity of the ZCB. In this work the two maturities are assumed to be equal for sake of simplicity and with no loss of generality.

Notice that both  $\Psi$  and  $V$  map  $[0, T] \times \mathcal{H}_w$  into  $\mathbb{R}$ . Moreover  $\Psi$  is a non-negative, uniformly bounded function with

$$\sup_{(t,h) \in [0,T] \times \mathcal{H}_w} \Psi(t, h) \leq K < 1. \quad (3.8)$$

Important regularity properties of  $\Psi$  are described in the following

**Proposition 3.1.** *There exist constants  $C_1, C_2 > 0$  such that*

$$|\Psi(t, h) - \Psi(t, g)| \leq C_1 \|h - g\|_w, \quad g, h \in \mathcal{H}_w, \quad t \in [0, T], \quad (3.9)$$

$$|\Psi(s, h) - \Psi(t, h)| \leq C_2 \|h\|_w |t - s|, \quad h \in \mathcal{H}_w, \quad s, t \in [0, T], \quad s \leq t. \quad (3.10)$$

*Proof.* Define the function  $\zeta : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\zeta(x) := [K - e^x]^+.$$

It is not hard to check that the weak derivative of  $\zeta$  is given by

$$\zeta'(x) = \begin{cases} 0 & x \geq \ln K, \\ -e^x & x < \ln K. \end{cases}$$

It follows that  $\|\zeta'\|_{L^\infty(\mathbb{R})} \leq K < 1$  and hence (cf. for instance [7], Chapter 8, Proposition 8.4)

$$|\zeta(x) - \zeta(y)| \leq \|\zeta'\|_{L^\infty(\mathbb{R})} |x - y| \leq |x - y|. \quad (3.11)$$

Now define  $X := -\int_0^{T-t} h(x) dx$  and  $Y := -\int_0^{T-t} g(x) dx$ , then (3.5) and (3.11) give

$$\begin{aligned} |\Psi(t, h) - \Psi(t, g)| &= |[K - e^X]^+ - [K - e^Y]^+| \leq |X - Y| \\ &\leq \int_0^{T-t} |h(x) - g(x)| dx \leq T \sup_{x \in \mathbb{R}_+} |h(x) - g(x)| \leq C T \|h - g\|_w, \end{aligned}$$

where the last inequality uses the continuous injection (2.8).

To prove (3.10) take  $s \leq t$  and proceed as above to obtain

$$|\Psi(t, h) - \Psi(s, h)| \leq \int_{T-t}^{T-s} |h(x)| dx \leq \sup_{x \in \mathbb{R}_+} |h(x)| |t - s| \leq C \|h\|_w |t - s|. \quad (3.12)$$

□

The case of a stochastic discount factor is considered in the following

**Corollary 3.2.** *Let  $X$  and  $Y$  be two  $\mathcal{H}_w$ -valued stochastic processes. Then*

$$\begin{aligned} \sup_{t \in [0, T]} \left| e^{-\int_0^t [X_u(0)(\omega)]^+ du} \Psi(t, X_t(\omega)) - e^{-\int_0^t [Y_u(0)(\omega)]^+ du} \Psi(t, Y_t(\omega)) \right| \\ \leq (C_1 + K C T) \sup_{t \in [0, T]} \|X_t(\omega) - Y_t(\omega)\|_w, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega, \end{aligned} \quad (3.13)$$



and

$$\begin{aligned} & \left| e^{-\int_0^s [X_u(0)(\omega)]^+ du} \Psi(s, X_s(\omega)) - e^{-\int_0^t [X_u(0)(\omega)]^+ du} \Psi(t, X_t(\omega)) \right| \\ & \leq (K + C_2) \sup_{t \in [0, T]} \|X_t(\omega)\|_w |t - s| + C_1 \|X_t(\omega) - X_s(\omega)\|_w, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \end{aligned} \quad (3.14)$$

*Proof.* The same arguments as in the proof of Proposition 3.1, the bound (3.8) and the Lipschitz condition (3.9) give

$$\begin{aligned} & \sup_{t \in [0, T]} \left| e^{-\int_0^t [X_u(0)(\omega)]^+ du} \Psi(t, X_t(\omega)) - e^{-\int_0^t [Y_u(0)(\omega)]^+ du} \Psi(t, Y_t(\omega)) \right| \\ & \leq K \sup_{t \in [0, T]} \left| e^{-\int_0^t [X_u(0)(\omega)]^+ du} - e^{-\int_0^t [Y_u(0)(\omega)]^+ du} \right| + \sup_{t \in [0, T]} |\Psi(t, X_t(\omega)) - \Psi(t, Y_t(\omega))| \\ & \leq K \sup_{t \in [0, T]} \int_0^t |[X_u(0)(\omega)]^+ - [Y_u(0)(\omega)]^+| du + C_1 \sup_{t \in [0, T]} \|X_t(\omega) - Y_t(\omega)\|_w. \end{aligned}$$

In the last expression use the fact that

$$\begin{aligned} |[X_u(0)(\omega)]^+ - [Y_u(0)(\omega)]^+| & \leq |X_u(0)(\omega) - Y_u(0)(\omega)| \\ & \leq \sup_{z \in \mathbb{R}_+} |X_u(z)(\omega) - Y_u(z)(\omega)| \leq C \|X_u(\omega) - Y_u(\omega)\|_w, \end{aligned}$$

with  $C > 0$  given by (2.8), to obtain

$$\begin{aligned} & \sup_{t \in [0, T]} \left| e^{-\int_0^t [X_u(0)(\omega)]^+ du} \Psi(t, X_t(\omega)) - e^{-\int_0^t [Y_u(0)(\omega)]^+ du} \Psi(t, Y_t(\omega)) \right| \\ & \leq (C_1 + K C T) \sup_{t \in [0, T]} \|X_t(\omega) - Y_t(\omega)\|_w. \end{aligned}$$

That concludes the proof of (3.13). Similar arguments provide the inequality (3.14).  $\square$

We now find some regularity properties of the value function  $V$ . An important role is played by Corollary 3.2.

**Proposition 3.3.** *The value function  $V$  is non-negative, uniformly bounded with the same upper bound  $K$  of  $\Psi$ ; that is,*

$$\sup_{(t, h) \in [0, T] \times \mathcal{H}_w} V(t, h) \leq K < 1. \quad (3.15)$$

Moreover, there exists  $L_V > 0$  such that

$$|V(t, h) - V(t, g)| \leq L_V \|h - g\|_w, \quad h, g \in \mathcal{H}_w, t \in [0, T]. \quad (3.16)$$

*Proof.* The first claim is obvious. To show (3.16) take  $h, g \in \mathcal{H}_w$  and fix  $t \in [0, T]$ . Then (3.7) and (3.13) imply

$$\begin{aligned} V(t, h) - V(t, g) & \leq \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ D^+(t, \tau; r_{\tau}^{t, h}(0)) \Psi(\tau, r_{\tau}^{t, h}) - D^+(t, \tau; r_{\tau}^{t, g}(0)) \Psi(\tau, r_{\tau}^{t, g}) \right\} \\ & \leq \mathbb{E} \left\{ \sup_{t \leq s \leq T} \left| D^+(t, \tau; r_{\tau}^{t, h}(0)) \Psi(\tau, r_{\tau}^{t, h}) - D^+(t, \tau; r_{\tau}^{t, g}(0)) \Psi(\tau, r_{\tau}^{t, g}) \right| \right\} \\ & \leq L_1 \mathbb{E} \left\{ \sup_{t \leq s \leq T} \|r_s^{t, h} - r_s^{t, g}\|_w \right\}, \end{aligned}$$



where  $L_1 := C_1 + CKT$ . Now invert the order and repeat the same arguments for  $V(t, g) - V(t, h)$ . In conclusion one obtains

$$|V(t, h) - V(t, g)| \leq L_1 \mathbb{E} \left\{ \sup_{t \leq s \leq T} \|r_s^{t, h} - r_s^{t, g}\|_w \right\}.$$

Denote by  $r^h$  the unique solution of (2.4) starting from  $h$  at time zero and similarly for  $r^g$ . The coefficients in (2.4) are time-homogeneous, hence (cf. (2.14))

$$\begin{aligned} \mathbb{E} \left\{ \sup_{t \leq s \leq T} \|r_s^{t, h} - r_s^{t, g}\|_w \right\} &= \mathbb{E} \left\{ \sup_{0 \leq s \leq T-t} \|r_s^h - r_s^g\|_w \right\} \\ &\leq \mathbb{E} \left\{ \sup_{0 \leq s \leq T} \|r_s^h - r_s^g\|_w \right\} \leq C_{1, T} \|h - g\|_w. \end{aligned}$$

Now (3.16) follows with  $L_V = L_1 C_{1, T}$ .  $\square$

It is clear that pricing the option amounts to study the value function  $V$  of an optimal stopping problem (cf. (3.6)) for a Hilbert space-valued diffusion. In this paper the function  $V$  will be characterized through an infinite-dimensional variational problem that will be obtained by suitably extending the recent results in [8].

## 4 Preliminary smoothing of the gain function

The pricing problem (3.6) is an optimal stopping problem involving gain function  $\Psi$  (cf. (3.5)) which is not smooth enough to allow a straightforward application of the results in [8]. In fact in [8] the Authors deal with a smooth gain function and no stochastic discount factor. They prove that the value function is a solution, in a suitable sense, of an infinite-dimensional variational problem obtained by solving a sequence of finite-dimensional ones. In the present case it is natural to tackle problem (3.6) by considering a regularized version of  $\Psi$ . For that we now introduce appropriate infinite-dimensional Sobolev spaces.

### 4.1 Gaussian measure and Sobolev spaces

Recall the operator  $Q$  of Definition 2.8. Define the centered Gaussian measure  $\mu$  with covariance operator  $Q$  (cf. [5], [11], [13]); that is, the restriction of the infinite product measure

$$\mu(dx) = \prod_{i=1}^{\infty} \frac{1}{\sqrt{2\pi\lambda_i}} e^{-\frac{x_i^2}{2\lambda_i}} dx_i, \quad (4.1)$$

to vectors  $x \in \ell_2$ . Let  $1 \leq p < \infty$  and consider  $f \in C_b^1(\mathcal{H}_w; \mathbb{R})$ . Define the  $L^p(\mathcal{H}_w, \mu)$  norm by

$$\|f\|_{L^p(\mathcal{H}_w, \mu)}^p := \int_{\mathcal{H}_w} |f(x)|^p \mu(dx). \quad (4.2)$$

If  $Df : \mathcal{H}_w \rightarrow \mathcal{H}_w^*$  is the Frechét derivative of  $f$  and  $\mathcal{H}_w$  is identified with its dual, then the  $L^2(\mathcal{H}_w, \mu; \mathcal{H}_w)$  norm of  $Df$  is defined as

$$\|Df\|_{L^2(\mathcal{H}_w, \mu; \mathcal{H}_w)}^2 := \int_{\mathcal{H}_w} \|Df(x)\|_w^2 \mu(dx). \quad (4.3)$$

Let  $\overline{D}$  denote the closure of  $D$  in  $L^2(\mathcal{H}_w, \mu)$  (cf. [11]) and let  $W^{1,2}(\mathcal{H}_w, \mu)$  be the Sobolev space defined by

$$W^{1,2}(\mathcal{H}_w, \mu) := \{f : f \in L^2(\mathcal{H}_w, \mu), \overline{D}f \in L^2(\mathcal{H}_w, \mu; \mathcal{H}_w)\}.$$

Notice however that in what follows derivatives are mostly generalized derivatives, hence there is no ambiguity in using  $D$  rather than  $\overline{D}$ .

**Remark 4.1.** For  $n \in \mathbb{N}$  the finite dimensional counterpart of  $\mu$ ,  $L^p(\mathcal{H}_w, \mu)$  and  $L^2(\mathcal{H}_w, \mu; \mathcal{H}_w)$  are  $\mu_n(dx) := \prod_{i=1}^n (2\pi\lambda_i)^{-\frac{1}{2}} e^{-\frac{x_i^2}{2\lambda_i}} dx_i$ ,  $L^p(\mathbb{R}^n, \mu_n)$  and  $L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)$ , respectively. Therefore if  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , then

$$\|f\|_{L^p(\mathcal{H}_w, \mu)}^p = \int_{\mathcal{H}_w} |f(x)|^p \mu(dx) = \int_{\mathbb{R}^n} |f(x)|^p \mu_n(dx) =: \|f\|_{L^p(\mathbb{R}^n, \mu_n)}^p$$

and

$$\|Df\|_{L^2(\mathcal{H}_w, \mu; \mathcal{H}_w)}^2 = \int_{\mathcal{H}_w} \|Df(x)\|_w^2 \mu(dx) = \int_{\mathbb{R}^n} \|Df(x)\|_{\mathbb{R}^n}^2 \mu_n(dx) =: \|Df\|_{L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)}^2.$$

The next proposition provides useful bounds on the gain function  $\Psi$  and its proof may be found in Appendix A.

**Proposition 4.2.** *There exist a positive constant  $C_\Psi$  such that the following estimates hold,*

$$\sup_{t \in [0, T]} \|\Psi(t)\|_{W^{1,2}(\mathcal{H}_w, \mu)} < C_\Psi \quad (4.4)$$

and

$$\int_0^T \left\| \frac{\partial \Psi}{\partial t}(t) \right\|_{L^2(\mathcal{H}_w, \mu)}^2 dt < C_\Psi. \quad (4.5)$$

Proposition 3.3 and arguments similar to those employed in the proof of Proposition 4.2 provide the following corollary for the value function  $V$  of problem (3.6).

**Corollary 4.3.** *There exists a positive constant  $C_V$  such that*

$$\sup_{t \in [0, T]} \|V(t)\|_{W^{1,2}(\mathcal{H}_w, \mu)} < C_V. \quad (4.6)$$

## 4.2 Smoothing the gain function

The smoothing procedure we introduce in this section will be obtained as a slight generalization of that used in [28], Chapter 4, Lemma 4.1. Define the family  $(\Phi_t)_{t \in [0, T]} \subset \mathcal{H}_w^*$  by

$$\Phi_t(h) := - \int_0^{T-t} h(x) dx, \quad h \in \mathcal{H}_w, \quad t \in [0, T]. \quad (4.7)$$

By continuous injection (2.8) follows

$$|\Phi_t(h)| \leq C_T \|h\|_w, \quad h \in \mathcal{H}_w,$$

for a positive constant  $C_T$ , i.e.  $(\Phi_t)_{t \in [0, T]} \subset \mathcal{H}_w^*$  is a bounded subset of  $\mathcal{H}_w^*$  with bound  $C_T$ .

Let  $C_b^{1,2}([0, T] \times \mathcal{H}_w)$  be the set of bounded continuous functions which are continuously differentiable once with respect to time and twice with respect to the space variable (in the Frechét sense) with bounded derivatives.

**Proposition 4.4.** *There exists a sequence  $(\Psi_k)_{k \in \mathbb{N}} \subset C_b^{1,2}([0, T] \times \mathcal{H}_w)$  satisfying (3.9) and (3.10), such that*

$$\Psi_k \rightarrow \Psi \quad \text{as } k \rightarrow \infty \quad \text{uniformly on } [0, T] \times \mathcal{H}_w, \quad (4.8)$$

and

$$\Psi_k \rightarrow \Psi \quad \text{as } k \rightarrow \infty \quad \text{in } L^2(0, T; L^2(\mathcal{H}_w, \mu)), \quad (4.9)$$

$$D\Psi_k \rightarrow D\Psi \quad \text{as } k \rightarrow \infty \quad \text{in } L^2(0, T; L^2(\mathcal{H}_w, \mu; \mathcal{H}_w)). \quad (4.10)$$

*Proof.* The gain function  $\Psi$  in (3.5) is the composition of  $f : [0, T] \times \mathcal{H}_w \rightarrow \mathbb{R}$ , with

$$f(t, h) := K - e^{\Phi_t(h)},$$

and  $g : \mathbb{R} \rightarrow \mathbb{R}_+$ , with  $g(z) := [z]^+$ .

Define  $I := Im(f) = (-\infty, K)$  and notice that  $g|_I : (-\infty, K) \rightarrow [0, K)$ , where  $g|_I$  is the restriction of  $g$  to the domain  $I$ . Let  $C_c^\infty(I)$  be the set of functions with compact support on  $I$  and continuously differentiable infinitely many times. Define the standard mollifiers  $(\rho_k)_{k \in \mathbb{N}} \subset C_c^\infty(I)$  and consider the mollified sequence  $(g_k)_{k \in \mathbb{N}} \subset C_c^\infty(I)$ , where  $g_k := \rho_k \star g$  (cf. [7], Chapter 4). Since  $g \in W^{1,p}(I)$  for all  $1 \leq p \leq \infty$ , then  $g_k \rightarrow g$  in  $W^{1,p}(I)$ ,  $1 \leq p < \infty$  as  $k \rightarrow \infty$ . It is well known that  $g'_k = \rho_k \star g'$ , where  $g'$  represents the weak derivative of  $g$ , moreover  $g_k \rightarrow g$  and  $g'_k \rightarrow g'$  pointwise as  $k \rightarrow \infty$ . The convergence is also locally uniform on every compact subset of  $\mathbb{R}$ , i.e.  $\|g_k - g\|_{L^\infty(\bar{I})} \rightarrow 0$ , as  $k \rightarrow \infty$  for any compact  $\bar{I} \subset \mathbb{R}$ . It is not hard to prove that

$$\|g_k\|_{L^\infty(I)} \leq \|g\|_{L^\infty(I)} = K \quad \text{and} \quad \|g'_k\|_{L^\infty(I)} \leq \|g'\|_{L^\infty(I)} = 1, \quad (4.11)$$

since  $g$  and its weak derivative  $g'$  are both uniformly bounded on  $I$ .

Notice that  $f \in C_b^{1,2}([0, T] \times \mathcal{H}_w)$  and therefore<sup>2</sup>

$$\frac{\partial(g_k \circ f)}{\partial t}(t, h) = g'_k(f(t, h)) \frac{\partial f}{\partial t}(t, h), \quad (4.12)$$

$$D(g_k \circ f)(t, h) = g'_k(f(t, h)) Df(t, h). \quad (4.13)$$

Using (4.11), the dominated convergence theorem and pointwise convergence of  $g_k$  and  $g'_k$ , give

$$g_k \circ f \rightarrow g \circ f \quad \text{in } L^2(0, T; L^2(\mathcal{H}_w, \mu)),$$

$$D(g_k \circ f) \rightarrow g'(f) Df \quad \text{in } L^2(0, T; L^2(\mathcal{H}_w, \mu; \mathcal{H}_w)).$$

Recall that  $D = \overline{D}$  is closed in  $L^2(\mathcal{H}_w, \mu)$  and hence  $D(g \circ f) = g'(f) Df$ .

If we set  $\Psi_k := g_k \circ f$ , then  $\Psi_k$  fulfils (3.9), (3.10) and (4.9) and (4.10) hold. Now the Lipschitz continuity (3.9), the mollifiers' properties and the obvious fact that  $|(x-y)^+ - (x)^+| \leq |y|$ , imply

$$\begin{aligned} |g_k(f(t, h)) - g(f(t, h))| &\leq \int_{\mathbb{R}} \rho_k(y) |g(f(t, h) - y) - g(f(t, h))| dy \leq \int_{\mathbb{R}} \rho_k(y) |y| dy \\ &= \int_{[-\frac{1}{k}, \frac{1}{k}]} \rho_k(y) |y| dy \leq \frac{1}{k}. \end{aligned} \quad (4.14)$$

The uniform convergence of  $\Psi_k$  to  $\Psi$  as  $k \rightarrow \infty$  follows from (4.14) since

$$\sup_{(t, h) \in [0, T] \times \mathcal{H}_w} |\Psi_k(t, h) - \Psi(t, h)| \leq \frac{1}{k}. \quad (4.15)$$

□

We now associate an optimal stopping problem to each smooth function  $\Psi_k$  and we denote by  $V_k$  its value function. That is, we set

$$V_k(t, x) := \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ e^{-\int_t^\tau [r_s^{t,r}(0)]^+ ds} \Psi_k(\tau, r_\tau^{t,r}) \right\} \quad (4.16)$$

<sup>2</sup>Differentiability with respect to time is obvious. Since  $\mathcal{H}_w$  is identified to its dual then  $\Phi_t \in \mathcal{H}_w$  with coordinates  $(\Phi_t^1, \Phi_t^2, \dots)$  and  $\|\Phi_t\|_w^2 = \sum_{i=1}^\infty (\Phi_t^i)^2$ . Hence  $\|Df(t, h)\|_w^2 = \sum_{i=1}^\infty (D_i f(t, h))^2 = e^{2\Phi_t(h)} \sum_{i=1}^\infty (\Phi_t^i)^2 = e^{2\Phi_t(h)} \|\Phi_t\|_w^2$ . Similarly  $\langle D^2 f(t, h) u, v \rangle \leq e^{\Phi_t(h)} \|\Phi_t\|_w^2 \|u\|_w \|v\|_w$ .

with  $r^{t,r}$  given by (2.4). It is not hard to verify that  $V_k$  has the same regularity properties as  $V$  (cf. Proposition 3.3). We prove that  $V_k$  is in fact an approximation of the value function  $V$  of the original problem (3.6).

**Proposition 4.5.**  $V_k \rightarrow V$  uniformly on  $[0, T] \times \mathcal{H}_w$ ; that is,

$$\lim_{k \rightarrow \infty} \sup_{(t,h) \in [0,T] \times \mathcal{H}_w} |V_k(t, h) - V(t, h)| = 0. \quad (4.17)$$

*Proof.* Fix  $(t, h) \in [0, T] \times \mathcal{H}_w$ , then

$$\begin{aligned} V_k(t, h) - V(t, h) &\leq \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ e^{-\int_t^\tau [r_s^{t,h}(0)]^+ ds} \left( \Psi_k(\tau, r_\tau^{t,h}) - \Psi(\tau, r_\tau^{t,h}) \right) \right\} \\ &\leq \mathbb{E} \left\{ \sup_{t \leq u \leq T} \left| \Psi_k(u, r_u^{t,h}) - \Psi(u, r_u^{t,h}) \right| \right\} \leq \frac{1}{k} \end{aligned}$$

by (4.15). The same holds for  $V(t, h) - V_k(t, h)$  and hence (4.17) follows.  $\square$

Notice that the approximating optimal stopping problem (4.16) may be characterized through variational methods as in [8], apart for minimal adjustments. Once  $V_k$  is found to be a solution of a suitable infinite-dimensional variational problem and the corresponding optimal stopping time is obtained, taking the limit as  $k \rightarrow \infty$  will lead to the variational formulation for the price function  $V$ . Then the optimal stopping time will be obtained by probabilistic arguments. Nevertheless, given the regularity of  $\Psi$ , we prefer to take limits of variational problems slightly weaker than those in [8].

## 5 An approximating variational inequality

A variational formulation of problem (4.16) is obtained by reducing the optimal stopping problem to a finite-dimensional setting. This is accomplished in two steps: first we make a Yosida approximation of the unbounded operator  $A$  in (2.4) by bounded operators  $A_\alpha$ , then we reduce the SDE itself to a finite dimensional one. At each step a corresponding optimal stopping problem is studied. In order to proceed with this algorithm the SDE (2.4) must live in a larger probability space. In particular, there is no loss of generality by assuming that  $W := (W^0, W^1, W^2, \dots)$  is an infinite dimensional standard Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P})$  and that the Brownian motion  $B$  of (2.4) coincides with its first component, i.e. we set  $W^0 = B$ . The original filtration can be replaced by the filtration generated by  $W$ , again denoted by  $\{\mathcal{F}_t, t \geq 0\}$  and completed by the null sets.

In this new setting all the arguments of the previous sections still hold and the pricing problem keeps the same form. In the next two sections we outline both the Yosida approximation and the finite-dimensional one. Full details may be found in [8].

### 5.1 Yosida approximation

The Yosida approximation of the unbounded linear operator  $A$  may be introduced without any further assumption and it is defined by  $A_\alpha := \alpha A(\alpha I - A)^{-1}$ , for  $\alpha > 0$  (cf. [33]). Since  $A_\alpha$  is a bounded linear operator, the corresponding SDE

$$\begin{cases} dr_t^{(\alpha)h} = [A_\alpha r_t^{(\alpha)h} + F_\sigma(r_t^{(\alpha)h})]dt + \sigma(r_t^{(\alpha)h})dW_t^0, & t \in [0, T], \\ r_0^{(\alpha)h} = h, \end{cases} \quad (5.1)$$

admits a unique strong solution  $r^{(\alpha)h}$ . That is,

$$r_t^{(\alpha)h} = h + \int_0^t [A_\alpha r_s^{(\alpha)h} + F_\sigma(r_s^{(\alpha)h})] ds + \int_0^t \sigma(r_s^{(\alpha)h}) dW_s^0, \quad t \in [0, T], \mathbb{P}\text{-a.s.}$$

For each  $\alpha > 0$ , the notations  $r^{(\alpha)h}$  and  $r^{(\alpha)t,h}$  are analogous to those used in Section 3. The following important convergence result is proven in [12], Proposition 7.5 and it is here recalled for completeness.

**Proposition 5.1.** *Let  $r^h$  be the unique mild solution of equation (2.4) with  $r_0^h = h$  and let  $r^{(\alpha)h}$  be the unique strong solution of equation (5.1). For  $p \geq 1$ , the following convergence holds*

$$\lim_{\alpha \rightarrow \infty} \mathbb{E} \left\{ \sup_{0 \leq t \leq T} \|r_t^{(\alpha)h} - r_t^h\|_w^p \right\} = 0, \quad h \in \mathcal{H}_w.$$

For  $k \in \mathbb{N}$  arbitrary but fixed, define  $V_{k,\alpha}$  as the value function of the optimal stopping problem corresponding to  $r^{(\alpha)h}$ , with regularized gain  $\Psi_k$  (cf. (4.16)),

$$V_{k,\alpha}(t, h) := \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ e^{-\int_t^\tau [r_s^{(\alpha)t,h}(0)]^+ ds} \Psi_k(\tau, r_\tau^{(\alpha)t,h}) \right\}. \quad (5.2)$$

Notice that  $V_{k,\alpha}$  satisfies (3.15) and (3.16) with the same constants. The convergence of  $V_{k,\alpha}$  to  $V_k$  (cf. (4.16)) as  $\alpha \rightarrow \infty$  holds both uniformly with respect to  $t$  and in a suitable  $L^p$ -norm.

**Theorem 5.2.** *The following convergence results hold,*

$$\lim_{\alpha \rightarrow \infty} \sup_{0 \leq t \leq T} |V_{k,\alpha}(t, h) - V_k(t, h)| = 0, \quad h \in \mathcal{H}_w, \quad (5.3)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^T \int_{\mathcal{H}_w} |V_{k,\alpha}(t, h) - V_k(t, h)|^p \mu(dh) dt = 0, \quad 1 \leq p < \infty, \quad (5.4)$$

where  $\mu$  is the Gaussian measure on the Hilbert space  $\mathcal{H}_w$ .

*Proof.* The arguments are similar to those used in the proof of Proposition 3.3. In fact the Lipschitz property of the gain function  $\Psi_k$  and the time-homogeneous character of the processes give

$$|V_{k,\alpha}(t, h) - V_k(t, h)| \leq L_V \mathbb{E} \left\{ \sup_{0 \leq s \leq T} \|r_s^{(\alpha)h} - r_s^h\|_w \right\}.$$

Since  $L_V$  is independent of  $t$ , the uniform convergence (5.3) follows from Proposition 5.1. To prove (5.4) it suffices to apply the dominated convergence theorem, since  $V_{k,\alpha}$  is uniformly bounded by  $K$ .  $\square$

**Remark 5.3.** *Notice that the convergence in (5.4) above holds for any finite measure on the Hilbert space  $\mathcal{H}_w$ .*

Standard results in Analysis provide the properties below.

**Theorem 5.4.** *If  $V_{k,\alpha} \in C_b([0, T] \times \mathcal{H}_w)$  for all  $\alpha > 0$ , then*

$$V_{k,\alpha} \rightarrow V_k \text{ as } \alpha \rightarrow \infty, \text{ uniformly on compact subsets } [0, T] \times \mathcal{K} \subset [0, T] \times \mathcal{H}_w. \quad (5.5)$$

Moreover  $V_k \in C_b([0, T] \times \mathcal{H}_w)$ .

*Proof.* We only outline the proof as a similar result is proved in [8], Theorem 3.3. Fix  $h \in \mathcal{H}_w$ , then (5.3) implies  $V_k(\cdot, h) \in C_b([0, T]; \mathbb{R})$ . For each  $\alpha > 0$  define

$$F_\alpha(h) := \sup_{t \in [0, T]} |V_{k, \alpha}(t, h) - V_k(t, h)|,$$

then  $F_\alpha(h) \rightarrow 0$  as  $\alpha \rightarrow \infty$  by (5.3). One may verify that the family  $(F_\alpha)_{\alpha > 0}$  is equi-bounded and equi-Lipschitz continuous, as (3.15) and (3.16) hold for both  $V_{k, \alpha}$  and  $V_k$ , hence  $V_{k, \alpha}$  converges uniformly to  $V_k$ , as  $\alpha \rightarrow \infty$ , on compact subsets  $[0, T] \times \mathcal{K}$  ([15], Theorem 7.5.6). It follows that  $V_k$  is continuous on any compact subset  $[0, T] \times \mathcal{K}$  (cf. [15], Theorem 7.2.1), being the uniform limit of bounded continuous functions. Finally  $V_k$  is continuous on  $[0, T] \times \mathcal{H}_w$  due to the Lipschitz property of  $h \mapsto V_k(t, h)$  uniformly with respect to  $t \in [0, T]$ .  $\square$

## 5.2 Finite dimensional reduction

For each  $n \in \mathbb{N}$  consider the finite dimensional subset  $\mathcal{H}_w^{(n)} := \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$  and the orthogonal projection operator  $P_n : \mathcal{H}_w \rightarrow \mathcal{H}_w^{(n)}$ . Approximate the diffusion coefficients of (5.1) by  $\sigma^{(n)} := (P_n \sigma) \circ P_n$ ,  $F_\sigma^{(n)} := (P_n F_\sigma) \circ P_n$  and  $A_{\alpha, n} := P_n A_\alpha P_n$ , respectively. Notice that  $A_{\alpha, n}$  is a bounded linear operator on  $\mathcal{H}_w^{(n)}$ . Define the process  $r^{(\alpha)h; n}$  as the unique strong solution of the SDE on  $\mathcal{H}_w^{(n)}$  given by

$$\begin{cases} dr_t^{(\alpha)h; n} = [A_{\alpha, n} r_t^{(\alpha)h; n} + F_\sigma^{(n)}(r_t^{(\alpha)h; n})]dt + \sigma^{(n)}(r_t^{(\alpha)h; n})dW_t^0 + \epsilon_n \sum_{i=1}^n \varphi_i dW_t^i, \\ r_0^{(\alpha)h; n} = P_n h =: h^{(n)}, \quad t \in [0, T], \end{cases} \quad (5.6)$$

where  $(\epsilon_n)_{n \in \mathbb{N}}$  is a sequence of positive numbers such that

$$\sqrt{n} \epsilon_n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5.7)$$

Obviously  $r^{(\alpha)h; n}$  lives in the finite dimensional subspace  $\mathcal{H}_w^{(n)}$  but it may still be seen as a solution in  $\mathcal{H}_w$ .

**Remark 5.5.** Notice that at each time  $t \in [0, S]$ ,  $r_t^{(\alpha)h; n}$  is not the projection of the process  $r_t^{(\alpha)h}$  on the finite dimensional subspace; in fact, a process with that property would not be Markovian. Hence  $r^{(\alpha)h; n}$  has to be considered as an auxiliary diffusion process which is used to approximate the original one.

**Proposition 5.6.** The following convergence

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ \sup_{t \in [0, S]} \left\| r_t^{(\alpha)h; n} - r_t^{(\alpha)h} \right\|_w^2 \right\} = 0, \quad (5.8)$$

holds uniformly with respect to  $h$  on compact subsets of  $\mathcal{H}_w$ .

*Proof.* The proof is based on standard arguments for  $L^p$ -estimates of SDE's strong solutions. It follows along the same lines as the proof of [8], Proposition 3.5. In fact, the only difference here is the presence of a non-linear drift term in (5.1) and (5.6) which, however, may be estimated by using (2.11) and (2.12).  $\square$

**Remark 5.7.** Notice that, for any starting time  $t \in [0, T]$ , the previous proposition and the arguments of its proof hold for  $r^{(\alpha)t, h; n}$  and  $r^{(\alpha)t, h}$  as well, thanks to the time-homogeneity of equations (5.1) and (5.6).

For  $n \geq 1$  define  $\Psi^{(n)} : [0, S] \times \mathcal{H}_w \rightarrow \mathbb{R}$  by

$$\Psi_k^{(n)}(t, h) := \Psi_k(t, P_n h) = \Psi_k(t, h^{(n)}) \quad (5.9)$$

(cf. (5.6)). Of course,  $P_n h^{(n)} = h^{(n)}$ , hence  $\Psi_k^{(n)}(t, \cdot) = \Psi_k(t, \cdot)$  on  $\mathcal{H}_w^{(n)}$ . However, in what follows it is convenient to use the notation  $\Psi_k^{(n)}$  since this is interpreted as a gain function on  $\mathcal{H}_w^{(n)}$ . The same arguments as in Appendix A, (A-3) show that  $\Psi_k^{(n)} \rightarrow \Psi_k$  as  $n \rightarrow \infty$  uniformly on every compact subset of  $[0, T] \times \mathcal{H}_w$ . Let  $V_{k,\alpha}^{(n)}$  be the value function of the optimal stopping problem

$$V_{k,\alpha}^{(n)}(t, h^{(n)}) := \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ e^{-\int_t^\tau [r_s^{(\alpha)t,h;n}(0)]^+ ds} \Psi_k^{(n)}(\tau, r_\tau^{(\alpha)t,h;n}) \right\}, \quad (5.10)$$

of course,  $V_{k,\alpha}^{(n)}$  may also be seen as a function on  $[0, T] \times \mathbb{R}^n$ . Notice that, as for  $V_{k,\alpha}$ , again  $V_{k,\alpha}^{(n)}$  satisfies (3.15) and (3.16) with the same constants. The value function  $V_{k,\alpha}^{(n)}$  converges to  $V_{k,\alpha}$  of (5.2) as  $n \rightarrow \infty$ . In fact results similar to Theorem 5.2 and Theorem 5.4 hold.

**Theorem 5.8.** *The following convergence results hold,*

$$\lim_{n \rightarrow \infty} \sup_{(t,h) \in [0,T] \times \mathcal{K}} |V_{k,\alpha}^{(n)}(t, h^{(n)}) - V_{k,\alpha}(t, h)| = 0, \quad \mathcal{K} \subset \mathcal{H}_w, \quad \mathcal{K} \text{ compact}, \quad (5.11)$$

*i.e. the convergence is uniform on any compact subset  $[0, T] \times \mathcal{K}$ ;*

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\mathcal{H}_w} |V_{k,\alpha}^{(n)}(t, h^{(n)}) - V_{k,\alpha}(t, h)|^p \mu(dh) dt = 0, \quad 1 \leq p < \infty, \quad (5.12)$$

where  $\mu$  is the Gaussian measure on the Hilbert space  $\mathcal{H}_w$ .

*Proof.* The proof follows along the same lines as the proof of Theorem 5.2 since  $\Psi_k^{(n)}(t, r_s^{(\alpha)t,h;n}) = \Psi_k(t, r_s^{(\alpha)t,h;n})$  for  $s \geq t$ . Then the uniform convergence in Proposition 5.6 implies (5.11), and by dominated convergence we obtain (5.12).  $\square$

As a consequence

**Theorem 5.9.** *If  $V_{k,\alpha}^{(n)} \in C_b([0, T] \times \mathcal{H}_w^{(n)})$  for all  $n \geq 1$ , then  $V_{k,\alpha} \in C_b([0, T] \times \mathcal{H}_w)$ .*

*Proof.* Recall that  $(V_{k,\alpha}^{(n)}(t, h^{(n)}))_{n \in \mathbb{N}}$  is uniformly bounded (cf. Proposition 3.3) and (5.11) holds. Hence [15], Theorem 7.2.1 guarantees the continuity of  $V_{k,\alpha}$  on  $[0, T] \times \mathcal{K}$ . Arguments as in Theorem 5.4 provide the continuity on  $[0, T] \times \mathcal{H}_w$ .  $\square$

Notice that in [8] the Authors proved that  $V_{k,\alpha}^{(n)}$  is indeed continuous.

### 5.3 A variational inequality for $V_k$

As in [8] we aim to characterize  $V_{k,\alpha}^{(n)}$  as a solution of a suitable variational problem on  $[0, T] \times \mathbb{R}^n$  by means of a slight modification of standard results (cf. [3] for instance). An optimal stopping time is found to be the first time when  $V_{k,\alpha}^{(n)} = \Psi_k^{(n)}$  and the continuity of the value function is obtained by the Sobolev embedding theorem. Taking limits as  $n \rightarrow \infty$  and  $\alpha \rightarrow \infty$  in the finite-dimensional variational inequality and using Theorems 5.2 and 5.8 enable us to identify  $V_k$  as a solution of an infinite-dimensional variational inequality. Then Theorems 5.4 and 5.9 allow us to find an optimal stopping time in (4.16). We omit the details of these proofs and the reader may refer to [8] for further insights.



The infinitesimal generator  $\mathcal{L}$  of the diffusion  $r$  is defined for every  $g \in C_b^2(\mathcal{H}_w; \mathbb{R})$  with  $Dg$  taking values in  $D(A^*)$  by

$$\mathcal{L}g(h) = \frac{1}{2}Tr [\sigma\sigma^*(x)D^2g(h)] + \langle h, A^*Dg(h) \rangle_w + \langle F_\sigma(h), Dg(h) \rangle_w, \quad \text{for } h \in \mathcal{H}_w. \quad (5.13)$$

We need to give a meaning to the infinite dimensional obstacle problem

$$\begin{cases} \max \left[ \frac{\partial v}{\partial t} + \mathcal{L}v - [h(0)]^+ v, \Psi_k - v \right] (t, h) = 0, & (t, h) \in (0, T) \times \mathcal{H}_w, \\ v \geq \Psi_k \text{ on } [0, T] \times \mathcal{H}_w; & v(T, h) = \Psi_k(T, h), \quad h \in \mathcal{H}_w, \end{cases} \quad (5.14)$$

and to prove that  $V_k$  is a suitable solution of it. In order to do so, for the time being, we start by arguing heuristically. At the finite dimensional level one finds that  $V_{\alpha,k}^{(n)} \in L^2(0, T; W^{1,p}(\mathbb{R}^n, \mu_n))$ ,  $2 \leq p < +\infty$ , and solves a weak version of the obstacle problem

$$\begin{cases} \max \left[ \frac{\partial v}{\partial t} + \mathcal{L}_{\alpha,n}v - [h^{(n)}(0)]^+ v, \Psi_k^{(n)} - v \right] (t, h^{(n)}) = 0, & (t, h^{(n)}) \in (0, T) \times \mathbb{R}^n, \\ v \geq \Psi_k^{(n)} \text{ on } (0, T) \times \mathbb{R}^n; & v(T, h^{(n)}) = \Psi_k^{(n)}(T, h^{(n)}), \quad h^{(n)} \in \mathcal{H}_w^{(n)}, \end{cases} \quad (5.15)$$

where  $\mathcal{L}_{\alpha,n}$  is the infinitesimal generator of diffusion (5.6). In principle we could take the limits as  $n \rightarrow \infty$  and  $\alpha \rightarrow \infty$ , and eventually obtain an infinite dimensional variational inequality for  $V_k$ , provided we find some specific estimates for  $V_{\alpha,k}^{(n)}$  and for the coefficients of its variational inequality.

We start by finding a weaker formulation of (5.14). Take  $2 < p < \infty$  and define the space

$$\mathcal{V}^p := \{v \mid v \in L^p(\mathcal{H}_w, \mu) \text{ and } Dv \in L^2(\mathcal{H}_w, \mu)\}, \quad (5.16)$$

endowed with the norm

$$\|v\|_p := \|v\|_{L^p(\mathcal{H}_w, \mu)} + \|Dv\|_{L^2(\mathcal{H}_w, \mu)}. \quad (5.17)$$

It is not hard to see that  $\mathcal{V}^p$  is a separable Banach space. In the spirit of [3] we will associate the second order differential operator in (5.14) to a bilinear form on  $\mathcal{V}^p$ . Let us first analyze the second term on the right-hand side of (5.13). Let  $\lambda$  be the Lebesgue measure on  $[0, T]$ . Denote by  $W^{1,2}([0, T] \times \mathcal{H}_w, \lambda \times \mu)$  the subset of functions  $u \in L^2(0, T; W^{1,2}(\mathcal{H}_w, \mu))$  with

$$\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\mathcal{H}_w, \mu)). \quad (5.18)$$

The set

$$\mathcal{E}_A([0, T] \times \mathcal{H}_w) := \text{span}\{\mathcal{R}e(\phi_{\eta,g}), \mathcal{I}m(\phi_{\eta,g}), \phi_{\eta,g}(t, h) = e^{i\frac{2\pi}{T}\eta t + i\langle g, h \rangle_w}, (\eta, g) \in \mathbb{N} \times D(A^*)\} \quad (5.19)$$

is dense in both  $W^{1,2}([0, T] \times \mathcal{H}_w, \lambda \times \mu)$  and  $L^2(0, T; L^p(\mathcal{H}_w, \mu))$  (cf. [11], Chapter 10 and [13], Chapter 9). For every  $u \in \mathcal{E}_A([0, T] \times \mathcal{H}_w)$  it is easy to check that  $A^*Du \in L^2(0, T; L^2(\mathcal{H}_w, \mu))$ . Hence, for  $v \in L^2(0, T; \mathcal{V}^p)$ , we define

$$T_A(v, u) := \int_0^T \int_{\mathcal{H}_w} \langle h, A^*Du \rangle_w v \mu(dh) dt, \quad \text{for } u \in \mathcal{E}_A([0, T] \times \mathcal{H}_w). \quad (5.20)$$

By Assumption 2.9 it was shown in [8], Section 5.3 that

$$|T_A(v, u)| \leq C_{\mu,p} \|v\|_{L^2(0,T;\mathcal{V}^p)} \|u\|_{L^2(0,T;\mathcal{V}^p)}, \quad u \in \mathcal{E}_A([0, T] \times \mathcal{H}_w) \quad (5.21)$$

for all  $2 < p < \infty$  and a suitable constant  $C_{\mu,p} > 0$  depending only on  $p$  and  $\mu$ . As  $\mathcal{E}_A([0, T] \times \mathcal{H}_w)$  is dense in  $L^2(0, T; \mathcal{V}^p)$ ,  $T_A(v, \cdot)$  is extended to the whole space  $L^2(0, T; \mathcal{V}^p)$  by Hahn-Banach theorem and the extended functional is denoted by  $\bar{T}_A(v, \cdot)$ .

For  $u, v \in L^2(0, T; \mathcal{V}^p)$  define the bilinear form

$$\begin{aligned} \int_0^T a_\mu(u, v) dt &:= \frac{1}{2} \int_0^T \int_{\mathcal{H}_w} \langle \sigma \sigma^* Du, Dv \rangle_w \mu(dh) dt + \frac{1}{2} \int_0^T \int_{\mathcal{H}_w} \text{Tr}[D\sigma]_{\mathcal{H}_w} \langle \sigma, Du \rangle_w v \mu(dh) dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\mathcal{H}_w} \langle D\sigma \cdot \sigma - \sigma \sigma^* Q^{-1} h, Du \rangle_w v \mu(dh) dt - \bar{T}_A(v, u) \\ &\quad - \frac{1}{2} \int_0^T \int_{\mathcal{H}_w} \langle F_\sigma(h), Du \rangle_w v \mu(dh) dt + \frac{1}{2} \int_0^T \int_{\mathcal{H}_w} [h(0)]^+ u v \mu(dh) dt, \end{aligned} \quad (5.22)$$

where  $D\sigma \cdot \sigma$  denotes the action of  $D\sigma \in \mathcal{L}(\mathcal{H}_w)$  on the vector  $\sigma \in \mathcal{H}_w$ .

The following estimate holds.

**Theorem 5.10.** *For every  $4 \leq p < \infty$  there exists a constant  $C_{\mu,\gamma,p} > 0$ , depending on  $\mu$ ,  $p$  and the bounds of  $\gamma$  in Assumption 2.10, such that*

$$\int_0^T |a_\mu(u(t), v(t))| dt \leq C_{\mu,\gamma,p} \left( \int_0^T \|u(t)\|_p^2 dt \right)^{\frac{1}{2}} \left( \int_0^T \|v(t)\|_p^2 dt \right)^{\frac{1}{2}} \quad (5.23)$$

for all  $u, v \in L^2(0, T; \mathcal{V}^p)$ .

*Proof.* The proof easily follows from Assumption 2.10, bound (5.21) and the estimate

$$\begin{aligned} \left| \frac{1}{2} \int_0^T \int_{\mathcal{H}_w} [h(0)]^+ u v \mu(dh) dt \right| &\leq \frac{1}{2} \int_0^T \int_{\mathcal{H}_w} \sup_{x \in \mathbb{R}_+} |h(x)| |u(t, h)| |v(t, h)| \mu(dh) dt \\ &\leq \frac{C}{2} \int_0^T \left[ \left( \int_{\mathcal{H}_w} \|h\|_w^2 \mu(dh) \right)^{\frac{1}{2}} \left( \int_{\mathcal{H}_w} |u(t, h)|^2 |v(t, h)|^2 \mu(dh) \right)^{\frac{1}{2}} \right] dt \\ &\leq \frac{C}{2} (\text{Tr} Q)^{\frac{1}{2}} \int_0^T \|u(t)\|_4 \|v(t)\|_4 dt, \end{aligned}$$

where  $C > 0$  is as in (2.8). Notice however that the term in (5.22) involving  $F_\sigma(h)$  is estimated by using bound (2.12).  $\square$

**Remark 5.11.** *For functions  $u, v$  in  $C_b^{1,2}([0, T] \times \mathcal{H}_w)$  with  $Du$  taking values in  $D(A^*)$ , (5.22) is simply obtained by*

$$\int_0^T a_\mu(u, v) dt := - \int_0^T \int_{\mathcal{H}_w} \left( \mathcal{L}u - [h(0)]^+ u \right) v \mu(dh) dt. \quad (5.24)$$

In fact, for  $u_n := u \circ P_n$ ,  $v_n := v \circ P_n$  defined on  $n$ -dimensional subspaces of  $\mathcal{H}_w$  and  $\sigma^{(n)}$ ,  $F_\sigma^{(n)}$  as in (5.6), (5.24) follows by Green's formula and (5.13). Then, by taking the limit as  $n \rightarrow \infty$  and using dominated convergence one finds that (5.24) holds in general.

Denote by  $(\cdot, \cdot)_\mu$  the scalar product in  $L^2(\mathcal{H}_w, \mu)$  and define  $F_k \in L^2(0, T; \mathcal{V}^p)^*$  by

$$F_k(v) := \int_0^T \left( \frac{\partial \Psi_k}{\partial t}(t) + \frac{1}{2} \text{Tr}[\sigma \sigma^* D^2 \Psi_k] + \langle F_\sigma, D\Psi_k \rangle_w, v \right)_\mu dt + \bar{T}_A(v, \Psi_k), \quad \forall v \in L^2(0, T; \mathcal{V}^p). \quad (5.25)$$

Take  $4 \leq p < \infty$  and introduce the closed, convex set

$$\mathcal{K}_\mu^p := \{w : w \in L^2(0, T; \mathcal{V}^p), \frac{\partial w}{\partial t} \in L^2(0, T; L^2(\mathcal{H}, \mu)), w \geq 0 \text{ } \lambda \times \mu\text{-a.e. in } (0, T) \times \mathcal{H}\}, \quad (5.26)$$

then  $\mathcal{K}_\mu^p \subset C([0, T]; L^2(\mathcal{H}, \mu))$  (cf. [16], Section 5.9.2).

The next theorem was proved in [8] (cf. Theorem 6.11 and Theorem 6.13) and it is the main result of that paper. The extension to the present setting is straightforward given the convergence results in Propositions 5.1 and 5.6 for the approximating diffusions  $r^{(\alpha)t, h}$  and  $r^{(\alpha)t, h; n}$ , the convergence results in Theorems 5.2, 5.4, 5.8 and 5.9 for the value functions  $V_{k, \alpha}^{(n)}$  and  $V_{k, \alpha}$ , and given Theorem 5.10. The proof is omitted as it may be found in [8].

**Theorem 5.12.** *For every  $4 \leq p < \infty$  the function  $u_k := V_k - \Psi_k$  is a solution of the weak variational problem*

$$\left\{ \begin{array}{l} u \in L^2(0, T; \mathcal{V}^p); \quad u(T, h) = 0, \quad h \in \mathcal{H}_w; \quad u(t, h) \geq 0, \quad (t, h) \in [0, T] \times \mathcal{H}_w; \\ \int_0^T \left[ -\left( \frac{\partial v}{\partial t}, v - u \right)_\mu + a_\mu(u, v - u) \right] dt - F_k(v - u) + \frac{1}{2} \|v(T)\|_{L^2(\mathcal{H}_w, \mu)}^2 \geq 0 \end{array} \right. \quad (5.27)$$

*for all  $v \in \mathcal{K}_\mu^p$ .*

Moreover,  $u_k \in C_b([0, T] \times \mathcal{H}_w)$  and an optimal stopping time for  $V_k$  in (4.16) is

$$\tau_k^*(t, h) := \inf\{s \geq t : V_k(s, r_s^{t, h}) = \Psi_k(s, r_s^{t, h})\} \wedge T. \quad (5.28)$$

Also, the for any stopping time  $\tau \leq \tau_k^*(t, h)$ ,

$$V_k(t, h) = \mathbb{E} \left\{ e^{-\int_t^\tau [r_u^{t, h}(0)]^+ du} V_k(\tau, r_\tau^{t, h}) \right\}. \quad (5.29)$$

## 6 A variational formulation for $V$

Since  $V_k$  converges to  $V$  as  $k \rightarrow \infty$  (cf. Proposition 4.5), it is natural to expect that  $V$  might be a solution of a variational problem similar to (5.27). However, that will be more likely to happen when taking limits of weaker variational problems, due to the lack of higher regularity of  $\Psi$ . In particular, notice that for all  $v \in \mathcal{E}_A([0, T] \times \mathcal{H}_w)$ , we may write  $F_k(v)$  (cf. (6.1)) as

$$F_k(v) = \int_0^T \left( \frac{\partial \Psi_k}{\partial t}(t), v \right)_\mu dt - \int_0^T a_\mu(\Psi_k(t), v) dt =: \int_0^T \mathcal{A}_{k, \mu}(t; v) dt, \quad (6.1)$$

by using arguments as in Remark 5.11. For  $v \in L^2(0, T; \mathcal{V}^p)$  one may approximate  $v$  by a sequence  $(v_j)_{j \in \mathbb{N}} \subset \mathcal{E}_A([0, T] \times \mathcal{H}_w)$ . Then, by taking the limit as  $j \rightarrow \infty$  and using estimates as in (5.23) one finds that (6.1) also holds for all  $v \in L^2(0, T; \mathcal{V}^p)$ .

Similarly, a continuous linear functional associated to  $\Psi$  may be defined by

$$\int_0^T \mathcal{A}_\mu(t; v) dt := \int_0^T \left( \frac{\partial \Psi}{\partial t}(t), v \right)_\mu dt - \int_0^T a_\mu(\Psi(t), v) dt, \quad \forall v \in L^2(0, T; \mathcal{V}^p). \quad (6.2)$$

Then a simple application of Proposition 4.4 and Theorem 5.10 imply

$$\lim_{k \rightarrow \infty} \left\| \int_0^T [\mathcal{A}_{k, \mu}(t; \cdot) - \mathcal{A}_\mu(t; \cdot)] dt \right\|_{L^2(0, T; \mathcal{V}^p)^*} = 0. \quad (6.3)$$

We now obtain the main result of the paper.

**Theorem 6.1.** *For every  $4 \leq p < \infty$  the function  $\hat{u} := V - \Psi$  is a solution of the weak variational problem*

$$\left\{ \begin{array}{l} u \in L^2(0, T; \mathcal{V}^p); \quad u(T, h) = 0, \quad h \in \mathcal{H}_w; \quad u(t, h) \geq 0, \quad (t, h) \in [0, T] \times \mathcal{H}_w; \\ \int_0^T \left[ -\left(\frac{\partial v}{\partial t}, v - u\right)_\mu + a_\mu(u, v - u) - \mathcal{A}_\mu(t; v - u) \right] dt + \frac{1}{2} \|v(T)\|_{L^2(\mathcal{H}_w, \mu)}^2 \geq 0 \end{array} \right. \quad (6.4)$$

for all  $v \in \mathcal{K}_\mu^p$ .

Moreover  $u \in C_b([0, T] \times \mathcal{H}_w)$ .

*Proof.* The boundary conditions are clearly satisfied by  $\hat{u}$ , moreover the continuity of  $\hat{u}$  is a consequence of Proposition 3.1 and Proposition 4.5. It remains to prove that  $\hat{u}$  solves the inequality in (6.4). From Propositions 3.3, 4.2, 4.4 and Corollary 4.3 easily follows that  $u_k$  of Theorem 5.12 is bounded in  $L^2(0, T; \mathcal{V}^p)$  by a constant  $M_{\mu, p, T} > 0$ , uniformly with respect to  $k \in \mathbb{N}$ , for all  $4 \leq p < \infty$ . This implies that  $u_k$  converges to some  $\bar{u}$  as  $k \rightarrow \infty$  weakly in  $L^2(0, T; \mathcal{V}^p)$ . However  $u_k \rightarrow \hat{u}$  as  $k \rightarrow \infty$  in  $L^2(0, T; L^p(\mathcal{H}_w, \mu))$  for all  $4 \leq p < \infty$ , by Propositions 4.4 and 4.5. Therefore  $\bar{u} = \hat{u}$ .

Notice that, for all  $v \in \mathcal{K}_\mu^p$ , the function  $u_k$  of Theorem 5.12 satisfies

$$\int_0^T \left[ -\left(\frac{\partial v}{\partial t}, v - u_k\right)_\mu + a_\mu(u_k, v - u_k) - \mathcal{A}_{k, \mu}(t; v - u_k) \right] dt + \frac{1}{2} \|v(T)\|_{L^2(\mathcal{H}_w, \mu)}^2 \geq 0 \quad (6.5)$$

by (5.27) and (6.1). In order to take limits as  $k \rightarrow \infty$  in (6.5) we use

$$\int_0^T a_\mu(u_k, u_k) dt = \int_0^T a_\mu(u_k - \hat{u}, u_k - \hat{u}) dt + \int_0^T a_\mu(\hat{u}, u_k) dt + \int_0^T a_\mu(u_k - \hat{u}, \hat{u}) dt. \quad (6.6)$$

Also, by using the bound  $M_{\mu, p, T}$ , (5.21), (5.22) and estimates as in the proof of Theorem 5.10 we obtain

$$\begin{aligned} \int_0^T a_\mu(u_k - \hat{u}, u_k - \hat{u}) dt &\geq -C_{\mu, p} \|u_k - \hat{u}\|_{L^2(0, T; \mathcal{V}^p)} \|u_k - \hat{u}\|_{L^2(0, T; L^p(\mathcal{H}_w, \mu))} \\ &\geq -C'_{\mu, p, T} \|u_k - \hat{u}\|_{L^2(0, T; L^p(\mathcal{H}_w, \mu))}, \end{aligned} \quad (6.7)$$

where  $C'_{\mu, p, T} = 2C_{\mu, p} M_{\mu, p, T}$ . Now we use (6.3), (6.7), (6.6) together with the convergence properties of  $u_k$  to obtain, in the limit,

$$\int_0^T \left[ -\left(\frac{\partial v}{\partial t}, v - \hat{u}\right)_\mu + a_\mu(\hat{u}, v - \hat{u}) - \mathcal{A}_\mu(t; v - \hat{u}) \right] dt + \frac{1}{2} \|v(T)\|_{L^2(\mathcal{H}_w, \mu)}^2 \geq 0. \quad (6.8)$$

□

We now show that the stopping time

$$\tau^*(t, h) := \inf\{s \geq t : V(s, r_s^{t, h}) = \Psi(s, r_s^{t, h})\} \wedge T \quad (6.9)$$

is optimal for  $V$  in (3.6). For that we need the next Lemma, whose proof follows along the lines of arguments adopted in [3], Chapter 3, Section 3, Theorem 3.7 (cf. in particular p. 322).

**Lemma 6.2.** *Let  $\tau_k^*(t, h)$  be as in (5.28) and let  $\tau^*(t, h)$  be as in (6.9). Then*

$$\lim_{k \rightarrow \infty} \tau_k^*(t, h) \wedge \tau^*(t, h) = \tau^*(t, h), \quad \mathbb{P}\text{-a.e.} \quad (6.10)$$

*Proof.* For simplicity we consider the diffusion  $r^h$  that starts at time zero from  $h$ . There is no loss of generality as all results below hold for arbitrary initial time  $t$ . We set  $\tau_k^* := \tau_k^*(0, h)$  and  $\tau^* := \tau^*(0, h)$ . By Theorem 5.12,  $\tau_k^*$  is optimal for the  $k$ -th regularized problem. The limit (6.10) is trivial for those  $\omega \in \Omega$  such that  $\tau^*(\omega) = 0$ . Set  $\Omega_0 := \{\omega \in \Omega : \tau^*(\omega) > 0\}$ . Fix  $\omega \in \Omega_0$  and take  $\delta < \tau^*(\omega)$ . Then, for  $t \in [0, \tau^*(\omega) - \delta]$ ,

$$V(t, r_t^h(\omega)) > \Psi(t, r_t^h(\omega)).$$

Since  $t \mapsto r_t^h(\omega)$  is continuous, the continuous map  $t \mapsto V(t, r_t^h(\omega)) - \Psi(t, r_t^h(\omega))$  attains its minimum on  $[0, \tau^*(\omega) - \delta]$ ; that is, there exists  $\eta(\delta, \omega) > 0$  such that

$$\eta(\delta, \omega) := \min\{V(t, r_t^h(\omega)) - \Psi(t, r_t^h(\omega)), t \in [0, \tau^*(\omega) - \delta]\}$$

and

$$V(t, r_t^h(\omega)) \geq \Psi(t, r_t^h(\omega)) + \eta(\delta, \omega), \quad \text{for all } t \in [0, \tau^*(\omega) - \delta].$$

Recall that  $\Psi_k \rightarrow \Psi$  and  $V_k \rightarrow V$  uniformly on  $[0, T] \times \mathcal{H}_w$  (cf. Propositions 4.4 and 4.5), therefore there exists  $N_\eta = N(\eta(\delta, \omega)) \in \mathbb{N}$  large enough and such that

$$V_k(t, r_t^h(\omega)) > \Psi_k(t, r_t^h(\omega)), \quad \text{for all } t \in [0, \tau^*(\omega) - \delta] \text{ and } k \geq N_\eta.$$

It follows that  $\tau^*(\omega) - \delta < \tau_k^*(\omega)$  for all  $k \geq N_\eta$ . Notice that  $\eta(\delta, \omega) \rightarrow 0$  as  $\delta \rightarrow 0$  and hence  $N_\eta \rightarrow \infty$ . Therefore

$$\lim_{\substack{N_\eta \rightarrow \infty \\ k \geq N_\eta}} (\tau_k^* \wedge \tau^*)(\omega) = \tau^*(\omega).$$

Since  $\omega \in \Omega_0$  is arbitrary, (6.10) follows.  $\square$

**Theorem 6.3.** *An optimal stopping time for  $V$  in (3.6) is given by  $\tau^*(t, h)$  of (6.9).*

*Proof.* Set  $\tau^* = \tau^*(t, h)$  and  $\tau_k^* = \tau_k^*(t, h)$  and take  $\tau = \tau_k^* \wedge \tau^*$  in (5.29) to obtain

$$V_k(t, x) = \mathbb{E} \left\{ e^{-\int_t^{\tau_k^* \wedge \tau^*} [r_u^{t,h}(0)]^+ du} V_k(\tau_k^* \wedge \tau^*, r_{\tau_k^* \wedge \tau^*}^{t,h}) \right\}. \quad (6.11)$$

By Proposition 4.5, the left hand side of (6.11) converges to  $V(t, x)$  in the limit as  $k \rightarrow \infty$ . In order to show the convergence of the right hand side, notice that the following estimate holds,

$$\begin{aligned} & \left| \mathbb{E} \left\{ e^{-\int_t^{\tau_k^* \wedge \tau^*} [r_u^{t,h}(0)]^+ du} \left[ V_k(\tau_k^* \wedge \tau^*, r_{\tau_k^* \wedge \tau^*}^{t,h}) - V(\tau^*, r_{\tau^*}^{t,h}) \right] \right\} \right| \\ & + \left| \mathbb{E} \left\{ \left[ e^{-\int_t^{\tau_k^* \wedge \tau^*} [r_u^{t,h}(0)]^+ du} - e^{-\int_t^{\tau^*} [r_u^{t,h}(0)]^+ du} \right] V(\tau^*, r_{\tau^*}^{t,h}) \right\} \right| \\ & \leq \mathbb{E} \left\{ |V_k(\tau_k^* \wedge \tau^*, r_{\tau_k^* \wedge \tau^*}^{t,h}) - V(\tau_k^* \wedge \tau^*, r_{\tau_k^* \wedge \tau^*}^{t,h})| \right\} + \mathbb{E} \left\{ |V(\tau_k^* \wedge \tau^*, r_{\tau_k^* \wedge \tau^*}^{t,h}) - V(\tau^*, r_{\tau^*}^{t,h})| \right\} \\ & + K \mathbb{E} \left\{ \int_{\tau_k^* \wedge \tau^*}^{\tau^*} [r_u^{t,h}(0)]^+ du \right\}. \end{aligned} \quad (6.12)$$

When  $k \rightarrow \infty$ , the first term on the right-hand side of (6.12) converges to zero by Proposition 4.5, whereas the second and third ones converge to zero by the continuity of  $V$  and Lemma 6.2. Hence by taking limits in (6.11) we may conclude that

$$V(t, h) = \mathbb{E} \left\{ e^{-\int_t^{\tau^*} [r_u^{t,h}(0)]^+ du} V(\tau^*, r_{\tau^*}^{t,h}) \right\} = \mathbb{E} \left\{ e^{-\int_t^{\tau^*} [r_u^{t,h}(0)]^+ du} \Psi(\tau^*, r_{\tau^*}^{t,h}) \right\}$$

and the optimality of  $\tau^*$  follows.  $\square$

## A Properties of the gain function

In order to prove Proposition 4.2 we need some auxiliary results about the regularity of  $\Psi$ . Let  $\{\varphi_1, \varphi_2, \dots\}$  be a set of orthonormal basis functions of  $\mathcal{H}_w$  and for  $n \geq 1$  let  $P_n : \mathcal{H}_w \rightarrow \mathcal{H}_w$  be the projection map defined by

$$P_n h := \sum_{i=1}^n \langle h, \varphi_i \rangle_w \varphi_i, \quad (\text{A-1})$$

where  $\langle \cdot, \cdot \rangle_w$  is the scalar product in  $\mathcal{H}_w$ . Set  $h^{(n)} := P_n h$  and define  $\Psi^{(n)} : [0, T] \times \mathcal{H}_w \rightarrow \mathbb{R}$  by

$$\Psi^{(n)}(t, h) := \Psi(t, P_n h) = \Psi(t, h^{(n)}). \quad (\text{A-2})$$

Dini's Theorem (cf. [15], Theorem 7.2.2) and (3.9) give

$$\lim_{n \rightarrow \infty} \sup_{(t, h) \in [0, T] \times \mathcal{K}} |\Psi^{(n)}(t, h) - \Psi(t, h)| = 0, \quad \text{for every compact } \mathcal{K} \subset \mathcal{H}_w. \quad (\text{A-3})$$

We will show now that  $\Psi^{(n)}$  belongs to a suitable Sobolev space. The arguments of the proof are similar to those employed in the infinite dimensional case (cf. [11], Chapter 10).

**Lemma A.1.** *Let  $\mu$  be a centered Gaussian measure on  $\mathcal{H}_w$ . The sequence  $(\Psi^{(n)})_{n \in \mathbb{N}}$  is uniformly bounded in  $L^\infty(0, T; W^{1,2}(\mathcal{H}_w, \mu))$ ; that is there exists  $C_\Psi > 0$  such that*

$$\sup_{t \in [0, T]} \|\Psi^{(n)}(t)\|_{W^{1,2}(\mathcal{H}_w, \mu)} < C_\Psi \quad \text{for all } n \in \mathbb{N}. \quad (\text{A-4})$$

Moreover there exists  $C'_\Psi > 0$  such that

$$\int_0^T \left\| \frac{\partial \Psi^{(n)}}{\partial t}(t) \right\|_{L^2(\mathcal{H}_w, \mu)}^2 dt < C'_\Psi \quad \text{for all } n \in \mathbb{N}. \quad (\text{A-5})$$

*Proof.* The uniform bound  $K$  of (3.8) gives

$$\sup_{t \in [0, T]} \|\Psi^{(n)}(t)\|_{L^2(\mathcal{H}_w, \mu)} \leq K \mu(\mathcal{H}_w) = K, \quad \text{for } n \in \mathbb{N}.$$

Notice that  $\Psi^{(n)}$  is a function defined on  $[0, T] \times \mathbb{R}^n$ ; that is  $\Psi^{(n)}(t, h) \equiv \Psi^{(n)}(t, h_1, \dots, h_n)$  for  $h_i := \langle h, \varphi_i \rangle_w$ ,  $i = 1, \dots, n$ . Hence we may mollify  $\Psi^{(n)}$  by the standard mollifiers  $(\rho_k)_{k \in \mathbb{N}}$ . In fact, fix  $t \in [0, T]$  and define  $\Psi_k^{(n)}(t, \cdot) := \rho_k \star \Psi^{(n)}(t, \cdot)$ . Clearly the pointwise convergence holds,  $\Psi_k^{(n)}(t, h) \rightarrow \Psi^{(n)}(t, h)$  as  $k \rightarrow \infty$ , for  $h \in \mathcal{H}_w$  (cf. [7], Proposition 4.21) and

$$|\Psi_k^{(n)}(t, z)| = \left| \int_{\mathbb{R}^n} \rho_k(y) \Psi^{(n)}(t, z - y) dy \right| \leq K \quad \text{for } z := (h_1, \dots, h_n) \in \mathbb{R}^n$$

by (3.8). Hence  $\sup_{(t, h) \in [0, T] \times \mathcal{H}_w} |\Psi_k^{(n)}(t, h)| \leq K$  for all  $n \in \mathbb{N}$ . It now follows that  $\Psi_k^{(n)}(t, \cdot) \rightarrow \Psi^{(n)}(t, \cdot)$  as  $k \rightarrow \infty$  in  $L^q(\mathcal{H}_w, \mu)$ ,  $1 \leq q < \infty$ , by dominated convergence. Since the bound  $K$  is uniform in  $t$ , dominated convergence and pointwise convergence also imply  $\Psi_k^{(n)} \rightarrow \Psi^{(n)}$  as  $k \rightarrow \infty$  in  $L^2(0, T; L^q(\mathcal{H}_w, \mu))$ .

It is easy to see that the mollified functions  $\Psi_k^{(n)}$  are equi-Lipschitz in the space variable, uniformly with respect to  $t$ , with the same constant  $C_1$  (cf. (3.9)). Therefore  $\|D\Psi_k^{(n)}(t)\|_{L^2(\mathcal{H}_w, \mu; \mathcal{H}_w)} = \|D\Psi_k^{(n)}(t)\|_{L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)} \leq C_1 \mu_n(\mathbb{R}^n) = C_1$ , for all  $n, k$  and  $t$ , by Remark 4.1. We may conclude that

$$\|\Psi_k^{(n)}(t)\|_{L^2(\mathcal{H}_w, \mu)} + \|D\Psi_k^{(n)}(t)\|_{L^2(\mathcal{H}_w, \mu; \mathcal{H}_w)} \leq (K + C_1) \quad \text{for } n, k \in \mathbb{N}, \text{ and } t \in [0, T]. \quad (\text{A-6})$$

Now (A-6) guarantees that for each  $n \in \mathbb{N}$  and  $t \in [0, T]$  there exists a function  $\phi^{(n)}(t) \in W^{1,2}(\mathcal{H}_w, \mu)$  and a subsequence  $(\Psi_{k_j}^{(n)}(t))_{j \in \mathbb{N}}$  such that  $\Psi_{k_j}^{(n)}(t) \rightharpoonup \phi^{(n)}(t)$  in  $W^{1,2}(\mathcal{H}_w, \mu)$  as  $j \rightarrow \infty$  (cf. [7], Theorem 3.18). It must be  $\Psi^{(n)}(t) = \phi^{(n)}(t)$  by uniqueness of the limit, since  $\Psi_k^{(n)}(t) \rightarrow \Psi^{(n)}(t)$  as  $k \rightarrow \infty$  in  $L^q(\mathcal{H}_w, \mu)$  and hence  $\Psi^{(n)}(t) \in W^{1,2}(\mathcal{H}_w, \mu)$  for  $t \in [0, T]$ . The lower semi-continuity of the weak limit and (A-6) imply the estimate (A-4) with  $C_\Psi = K + C_1$ .

Similarly, to show (A-5) we may apply the same arguments as before and use the Lipschitz property of  $\Psi^{(n)}(\cdot, h)$ , for  $h \in \mathcal{H}_w$  fixed. This gives

$$\left\| \frac{\partial \Psi^{(n)}}{\partial t}(h) \right\|_{L^2([0, T])}^2 \leq 2C_2^2(1 + \|h\|_{\mathcal{H}_w}^2). \quad (\text{A-7})$$

Taking the integral in (A-7) with respect to  $\mu$  and applying Fubini's theorem gives (A-5).  $\square$

We are now ready to provide the main result of this Appendix.

*Proof of Proposition 4.2.* Take the projection  $\Psi^{(n)}$  of  $\Psi$  as in (A-2). The sequence  $(\Psi^{(n)}(t, \cdot))_{n \in \mathbb{N}}$  is bounded in  $W^{1,2}(\mathcal{H}_w, \mu)$  by Lemma A.1 uniformly in  $t \in [0, T]$ . Therefore for every  $t \in [0, T]$  there exists  $\Phi(t) \in W^{1,2}(\mathcal{H}_w, \mu)$  such that  $\Psi^{(n)}(t) \rightharpoonup \Phi(t)$  in  $W^{1,2}(\mathcal{H}_w, \mu)$  as  $n \rightarrow \infty$ . By (A-3) and dominated convergence  $\Psi^{(n)}(t) \rightarrow \Psi(t)$  in  $L^p(\mathcal{H}_w, \mu)$ ,  $1 \leq p < \infty$  as  $n \rightarrow \infty$  for all  $t \in [0, T]$ ; hence  $\Phi(t) = \Psi(t)$  for all  $t \in [0, T]$ . Moreover, since the bound in (A-4) is uniform with respect to  $n \in \mathbb{N}$  and  $t \in [0, T]$ , the lower semi-continuity of the weak convergence gives (4.4).

To prove (4.5) it suffices to adopt arguments as above and use bound (A-5).  $\square$

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